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# A THEOREM IN DETERMINANTS.

By Prof. E. O. Lovett, Princeton, N. J.

1. The expression

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 \\ a_4 & b_4 \end{vmatrix} \tag{1}$$

is obviously identically zero.

The identity corresponding to this in terms of determinants of the third order is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 & c_2 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix}$$

$$- \begin{vmatrix} a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \\ a_5 & b_5 & c_5 \end{vmatrix} \cdot \begin{vmatrix} a_3 & b_3 & c_3 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix} = 0. \tag{2}$$

The following is a generalization of these identities for determinants of the $n$th order, viz., the sum of the expressions

$$\begin{aligned} &|\,1, 2, 3, 4, \ldots, n\,| \,.\, |\,n+1, n+2, n+3, \ldots, 2n\,| \\ (-1)^n\,&|\,2, 3, 4, \ldots, n, n+1\,| \,.\, |\,1, n+2, n+3, \ldots, 2n\,| \\ &|\,3, 4, \ldots, n, n+1, 1\,| \,.\, |\,2, n+2, n+3, \ldots, 2n\,| \\ (-1)^n\,&|\,4, \ldots, n, n+1, 1, 2\,| \,.\, |\,3, n+2, n+3, \ldots, 2n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (-1)^n\,&|\,n+1, 1, 2, 3, \ldots, n-1\,| \,.\, |\,n, n+2, n+3, \ldots, 2n \end{aligned} \tag{3}$$

is identically zero, where for convenience we put

$$|\,a_1^{(1)}, a_2^{(2)}, a_3^{(3)}, \ldots, a_n^{(n)}\,| = |\,1, 2, 3, \ldots, n\,|. \tag{4}$$

The identity of this sum with zero is made clear by expanding the determinants with regard to the elements of their left-hand columns; then on

forming the products and summing, it appears that the coefficient of the general term is of the form

$$\begin{aligned} &|\,1, 2, 3, 4, \ldots, n-1\,|\,.\,|\,n, n+1, n+2, \ldots, 2(n-1)\,| \\ &+(-1)^{n-1}\,|\,2, 3, 4, \ldots, n-1, 1\,|\,.\,|\,1, n+1, n+2, \ldots, 2(n-1)\,| \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+(-1)^{n-1}\,|\,n, 1, 2, \ldots, n-2\,|\,.\,|\,n-1, n+1, n+2, \ldots, 2(n-1)\,|\,; \end{aligned} \tag{5}$$

treating this sum in the same manner the coefficient of its general term is found to be of the form (5) in which $n$ is again decreased by unity; this process carried through $n-2$ steps leads to a coefficient of the form

$$|\,1, 2\,|\,.\,|\,3, 4\,| + |\,2, 3\,|\,.\,|\,1, 4\,| + |\,3, 1\,|\,.\,|\,2, 4\,|\,, \tag{6}$$

which is (1) in a different notation and identically zero; therefore, *a fortiori*, is the sum of the expressions (3) identically zero. Hence, we have

$$\Sigma\,|\,\alpha, \beta, \gamma, \delta, \ldots, \mu\,|\,.\,|\,\nu, n+2, n+3, \ldots, 2n\,| = 0\,, \tag{7}$$

where

$$\alpha, \beta, \gamma, \delta, \ldots, \mu, \nu$$

takes successively the values of the sequence

$$1, 2, 3, 4, \ldots, n, n+1\,,$$

(or in fact, of any sequence of $n+1$ different indices) and of its cyclical permutations, and is always positive or alternately positive and negative, according as $n$ is even or odd.

2. By noting that the volume of a solid having $n+1$ vertices in a space of $n$ dimensions is equal to

$$\frac{1}{n!}\begin{vmatrix} x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}, & 1 \\ x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}, & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{(n+1)}, & x_2^{(n+1)}, & \ldots, & x_n^{(n+1)}, & 1 \end{vmatrix}, \tag{8}$$

$$x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}, \quad i = 1, 2, 3, \ldots, n+1\,, \tag{9}$$

being the coordinates of the $n+1$ vertices, the above theorems may be interpreted geometrically.

Consider a system of $2(n+1)$ points in a space of $n$ dimensions. These points form $\lfloor 2(n+1) / (\lfloor n+1)^2$ solids, each having $n+1$ vertices. Designating the points by the integers $1, 2, \ldots, n+1, n+2, \ldots, 2(n+1)$, and the volume of the solid whose $n+1$ vertices are $1, 2, 3, \ldots, n+1$ by $\{1, 2, 3, \ldots, n+1\}$, the theorem expressed by (7) becomes, in virtue of (8) and (5)

$$\{1, 2, \ldots, n, n+1\} \cdot \{n+2, n+3, \ldots, 2(n+1)\} \pm \{1, 2, \ldots, n, n+2\} \cdot \{n+3, n+4, \ldots, 2(n+1), n+1\}$$

$$+ \{1, 2, \ldots, n, n+3\} \cdot \{n+4, \ldots, 2(n+1), n+1, n+2\} \pm \{1, 2, \ldots, n, n+4\} \cdot \{n+5, \ldots, 2(n+1), n+1, n+2, n+3\}$$

$$+ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\pm \{1, 2, \ldots, n, 2(n+1)\} \cdot \{n+1, n+2, n+3, \ldots, 2n+1\} = 0, \qquad (10)$$

the upper or lower signs being taken according as $n$ is odd or even.

The volumes of these solids arrange themselves in pairs in $\lfloor 2(n+1) / 2(\lfloor n+1)^2$ products to satisfy $[2(n+1)/2(n+1)^2] - (n^2+1)$ independpendent identical relations of the form (10); the others are derived from (10) by permutation. The $n^2$ independent ratios of these binary products are the independent absolute invariants of the original system of $2(n+1)$ points by the general projective group of $n$ dimensional space; the identities of the form (10) thus furnish a complete generalization of Clifford's statement of the general theory of anharmonics for the first three dimensions.

It may be further remarked by way of illustration that the identity (10) gives the well known relation connecting the distances of four points 1, 2, 3, 4 on a right line, viz.,

$$12 \cdot 34 + 23 \cdot 14 + 31 \cdot 24 = 0,$$

and the five independent identities* of the form

$$123 \cdot 456 + 124 \cdot 365 + 125 \cdot 634 + 126 \cdot 543 = 0,$$

connecting the areas of triangles formed by six points 1, 2, 3, 4, 5, 6 in a plane.

* Clifford is in error both as to the statement and the number of these identities on page 3 of volume 2 of the *Proceedings of the London Mathematical Society*.

# ON THE EXPANSION OF AN ARBITRARY FUNCTION IN TERMS OF LAPLACE'S FUNCTIONS.

By Prof. W. H. Echols, Charlottesville, Va.

1. We recall the following theorem :

If $f(x)$ is a one-valued and continuous function that can be expanded in integral powers of $x - a$ throughout an interval $((\alpha, \beta))$, where $2a = \alpha + \beta$, then can $f(x)$ be expanded in the series

$$f(x) = A_0 + A_1\varphi_1(x) + A_2\varphi_2(x) + \ldots, \tag{1}$$

where $\varphi_r(x)$, $(r = 1, 2, \ldots)$, are rational integral functions of $x$ of degree $r$ constructed according to any assigned law and the coefficients $A_r$ are independent of $x$. The series (1) being convergent and representing $f(x)$ in $((\alpha, \beta))$.

A demonstration of this consists in designing a rational integral function, with the given $\varphi_r(x)$ functions,

$$A_0 + A_1\varphi_1(x) + \ldots + A_n\varphi_n(x), \tag{2}$$

which shall cut $f(x)$ at $a - a_0$ and at $n$ other arbitrarily chosen points, $a_1, \ldots, a_n$ in $((\alpha, \beta))$, and then making $a_1 = a_2 = \ldots = a_n = a$.

Thus if $R$ be the difference between $f(x)$ and (2) at any point $x$ in $((\alpha, \beta))$ we have

$$f(x) = \sum_0^n A_r\varphi_r(x) + R, \tag{3}$$

and in order that this difference shall be 0 at $a_1, a_2, \ldots, a_n$, we have

$$f(a_j) = \sum_0^n A_r a_j^r, \tag{4}$$

$j = 1, \ldots, n$. Eliminating the $A$'s from (3) and (4), the function (2) is designed. Thus

$$\frac{\begin{vmatrix} f(x), & 1, & \varphi_1(x), & \ldots, & \varphi(x) \\ f(a), & 1, & \varphi_1(a), & \ldots, & \varphi_n(a) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f(a_n), & 1, & \varphi_1(a_n), & \ldots, & \varphi_n(a_n) \end{vmatrix}}{\begin{vmatrix} 1, & \varphi_1(a), & \ldots, & \varphi_n(a_n) \end{vmatrix}} = R. \tag{5}$$

But the function $R$ is a one-valued and continuous function of $x$ which admits an indefinite number of one-valued and continuous derivatives in $((\alpha, \beta))$. Since it has the $n+1$ zeros $a_0, \ldots, a_n$, we have for the value of $R$, by a well known elementary theorem,

$$R = \frac{(x-a_0)\ldots(x-a_n)}{(n+1)!}\left[\frac{d^{n+1}R}{dx^{n+1}}\right]_{x=u},$$

where $\left[\frac{d^{n+1}R}{dx^{n+1}}\right]_{x=u}$, means the $(n+1)$th derivative of $R$ at some point $u$ between the greatest and least of the numbers $x, a_0, \ldots, a_n$. But since the degree of the rational integral functions in the first row is not greater than $n$, all their $(n+1)$th derivatives are zero, so that

$$R = \frac{(x-a_0)\ldots(x-a_n)}{(n+1)!} f^{n+1}(u),$$

which converges to the limit

$$R = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u),$$

when the $a$'s converge to the limit $a$, and now $u$ lies between $x$ and $a$. At the same time the limit of the left side of (5) is evaluated in the usual way by performing on the numerator and denominator the operation

$$\left[\frac{\partial}{\partial a_1}\right]^1_{a_1=a}\left[\frac{\partial}{\partial a_2}\right]^2_{a_2=a}\cdots\left[\frac{\partial}{\partial a_n}\right]^n_{a_n=a}.$$

Whence we have

$$\frac{\begin{vmatrix} f(x), & 1, & \varphi_1(x), & \ldots, & \varphi_n(x) \\ f(a), & 1, & \varphi_1(a), & \ldots, & \varphi_n(a) \\ f'(a), & 0, & \varphi_1'(a), & \ldots, & \varphi_n'(a) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n(a), & 0, & 0, & \ldots, & \varphi_n^n(a) \end{vmatrix}}{\varphi_1'(a)\,\varphi_2''(a)\ldots\varphi_n^n(a)} = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u),$$

or

$$f(x) - \sum_0^n A_r\varphi_r(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u).$$

By hypothesis the limit of the right side is zero, since this is the condition that $f(x)$ can be expanded in powers of $(x - \alpha)$, when $n = \infty$ for all values of $x$ in $((\alpha, \beta))$. By the above construction the coefficients $A_r$ are assigned numbers for any assigned value of the integer $r$, and we have*

$$f(x) = \sum_0^\infty A_r \varphi_r(x). \tag{6}$$

2. The function

$$\varphi_r(x) = \frac{d^r}{dx^r} \{(x - \alpha)(x - \beta)\}^r,$$

is a rational integral function of degree $r$, which we write $P_x^r$ for brevity. The function $f(x)$ being as before in § 1, we have

$$f(x) = \sum_0^\infty A_r P_x^r. \tag{7}$$

This function admits of integration between the limits $\alpha$ and $\beta$, and we can determine the coefficients $A_r$ in a new form as follows.

The function $P_x^r$ is such that its first $r$ integrals taken from $\alpha$ to $\beta$ are zero, for the parts independent of the arbitrary constants will contain some power of $(x - \alpha)(x - \beta)$ as a factor.

Write

$$\left\{ \int_\alpha^x dx \int_\alpha^x dx f(x) \right\}_{x=\beta} = F_2,$$

$$\left\{ \int_\alpha^x dx \int_\alpha^x dx P_x^r \right\}_{x=\beta} = P_2^r,$$

with obvious generalization.

Then integrating (7) successively between $\alpha$ and $\beta$, we have

$$F_1 = h_1 A_0,$$

$$F_2 = h_2 A_0 + P_2^1 A_1,$$

$$F_3 = h_3 A_0 + P_3^1 A_1 + P_3^2 A_2,$$

. . . . . .

* See Annals of Mathematics, Vol. VII, p. 41.

Wherein for brevity we write $h_r = (\beta - \alpha)^r/r!$. These equations serve to determine the $A_r$'s. Thus

$$A^{r-1} = (-1)^r \frac{\begin{vmatrix} F_1, & h_1, & 0 & 0 & \dots, & 0 \\ F_2, & h_2, & P_2^1, & 0 & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ F_r, & h_r, & P_r^1, & P_r^2, & \dots, & P_r^{r-2} \end{vmatrix}}{h_1 P_2^1 \dots P_r^{r-1}}.$$

An assigned number corresponding to an assigned $r$. This coefficient contains no derivatives of $f(x)$, but is constructed wholly with definite integrals of the functions involved. In particular

$$A_0 = \frac{1}{\beta - \alpha}\int_\alpha^\beta f(x)\,dx$$

is the mean value of $f(x)$ in $((\alpha, \beta))$. We propose to examine this expansion from an entirely independent point of view.

3. The successive integrals of a function $\psi(x)$ can always be expressed by a single integral as follows.

Let

$$\int \psi(x)\,dx = \psi_1(x),$$

then

$$D_x\psi_1(x) = \psi(x).$$

Generally, if

$$\left[\int_a^x dx\right]^r \psi(x) = \psi_r(x),$$

then

$$\left[\int_a^x dx\right]^{r-1} \psi(x) = D_x\psi_r(x) = \psi_{r-1}(x).$$

Now,

$$\int_a^x \psi(x)dx = \psi_1(x) - \psi_1(a),$$

$$\left[\int_a^x dx\right]^2 \psi(x) = \psi_2(x) - \psi_2(a) - (x - a)\,\psi_1(a),$$

and by an easy generalization,

$$\left[\int_a^x dx\right]^n \psi(x) = \psi_n(x) - \sum_0^{n-1} \frac{(x-a)^r}{r!}\, \psi_{n-r}(a)\,.$$

But

$$\frac{d}{da}\left\{\psi_n(x) - \sum_0^{n-1} \frac{(x-a)^r}{r!}\, \psi_{n-r}(a)\right\} = -\frac{(x-a)^{n-1}}{(n-1)!}\, \psi(a)\,.$$

Integrating between $a$ and $x$, we have

$$\psi_n(x) - \sum_0^{n-1} \frac{(x-a)^r}{r!}\, \psi_{n-r}(a) = \int_a^x \frac{(x-a)^{n-1}}{(n-1)!}\, \psi(a)\, da\,.$$

Since $x - a$ is positive for $x$ in $((a, \beta))$, this can be written

$$\psi(u) \int_a^x \frac{(x-a)^{n-1}}{(n-1)!}\, da = \frac{(x-a)^n}{n!}\, \psi(u)\,,$$

where $u$ is some number in the interval $(a, x)$.

4. If the first $n$ integrals of a function $\psi(x)$,

$$\psi_r(x) = \left[\int_a^x dx\right]^r \psi(x)\,,$$

$(r = 1, 2, \ldots, n)$ are zero when $x = \beta$, then the function $\psi(x)$ is 0 for $n$ distinct values of $x$ in $(a, \beta)$. For, $\psi_n(x) = 0$ when $x = a$ and $x = \beta$. Its derivative $\psi_{n-1}(x)$ is therefore zero for some value $u_1$ between $a$ and $\beta$. Therefore, since $\psi_{n-1}(x)$ is zero at $a$, $u_1$, $\beta$, its derivative $\psi_{n-2}(x)$ is zero for some value of $x$ in each of the intervals $(a, u_1)$, $(u_1, \beta)$. But $\psi_{n-2}(x)$ is also zero at $a$ and $\beta$, or vanishes four distinct times in $((a, \beta))$. Continuing, thus we find that $\psi_1(x)$ is zero at $a$ and $\beta$ and also at $n - 1$, distinct values between $a$ and $\beta$, or altogether $n + 1$ times in the interval $((a, \beta))$. Therefore the derivative $\psi'(x)$ of $\psi_1(x)$ vanishes $n$ distinct times in $(a, \beta)$.

5. Employing the notations of § 2, the function

$$\frac{\begin{vmatrix} f(x), & 1, & P_x^1, & P_x^2, & \ldots, & P_x^n \\ F_1, & h_1, & 0, & 0, & \ldots, & 0 \\ F_2, & h_2, & P_2^1, & 0, & \ldots, & 0 \\ F_3, & h_3, & P_3^1, & P_3^2, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ F_{n+1}, & h_{n+1}, & P_{n+1}^1, & P_{n+1}^2, & \ldots, & P_{n+1}^n \end{vmatrix}}{h_1 P_2^1 \ldots P_{n+1}^n}\,, \qquad (8)$$

is such that its first $n+1$ integrals between $\alpha$ and $x$ all vanish for $x = \beta$.* Therefore this function vanishes $n+1$ times between $\alpha$ and $\beta$ at, say, $u_1, \ldots, u_{n+1}$. It is therefore equal to

$$\frac{(x-u_1)\ldots(x-u_{n+1})}{(n+1)!} f^{n+1}(u),$$

where $u$ lies between the greatest and the least of the numbers $x, u_1, \ldots, u_{n+1}$. But

$$\begin{aligned} |(x-u_1)\ldots(x-u_{n+1})| &< |\beta-\alpha|^{n+1}, \\ &= \theta(\beta-\alpha)^{n+1}, \end{aligned}$$

where $-1 < \theta < +1$, for all values of $x$ in $((\alpha, \beta))$. Therefore, we have

$$f(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx + \sum_1^n A_r P_x^r + \theta\frac{(\beta-\alpha)^{n+1}}{(n+1)!} f^{n+1}(u). \tag{9}$$

If $f(x)$ can be expanded in powers of $(x-\alpha)$ up to and including $\beta$, the last term on the right has 0 for its limit when $n = \infty$ and the series

$$f(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx + \sum_1^\infty A_r P_x^r \tag{10}$$

is convergent and represents $f(x)$ for all values of $x$ in $((\alpha, \beta))$. In the series (10) the coefficients $A$ are independent of $x$ and of derivatives of $f(x)$, being constructed with definite integrals from $\alpha$ to $\beta$.

The test for the *convergence* of the series is unfortunately made to depend on the derivatives of $f(x)$. It is desirable that the equivalent of (8) should be evaluated in terms of a definite integral of $f(x)$. The great importance in physics of the expansion of an arbitrary function in terms of Laplace's functions is well known. It is analytically of the greatest importance that the expansion of a one-valued function that is continuous in an interval should be completely solved independent of the operation of differentiation, since there exist such functions whose derivatives are wholly indeterminate and therefore can play no part in the investigation of their properties.†

* As a matter of detail we observe that $P_x^n$ has its $n$ roots real and in $(\alpha, \beta)$, and also for $r < s$

$$P_{s+m}^{r+m} = P_s.$$

† Annals of Mathematics, Vol. 10, p. 17.

# ON THE RELATION BETWEEN CAUCHY'S NUMBERS AND BESSEL'S FUNCTIONS.

By Dr. Alexander S. Chessin, New York.

It is sometimes of advantage in Celestial Mechanics to transform formulas expressed by means of Cauchy's numbers into a form where Bessel's functions are the instrument of calculations and *vice versa*. It is therefore useful to know what relations there exist between Cauchy's numbers and Bessel's functions.

In a note on Cauchy's numbers by the author* it has been shown that

$$N_{-p,j,q} = \left[\frac{j}{n}\right] - \left[\frac{j}{n-1}\right]\left[\frac{q}{1}\right] + \left[\frac{j}{n-2}\right]\left[\frac{q}{2}\right] - \ldots + (-1)^n \left[\frac{q}{n}\right] \quad (1)$$

where $n = \frac{1}{2}(-p + j + q)$, and the symbol $\left[\frac{a}{b}\right]$ stands for

$$\frac{a\,(a-1)\ldots(a-b+1)}{b\,!}.$$

This formula gives for $j = 0$

$$N_{-p,0,q} = (-1)^n \left[\frac{q}{n}\right] = (-1)^n \frac{q\,!}{n\,!\,(q-n)\,!} = (-1)^n \frac{q\,!}{n\,!\,(n+p)\,!}. \quad (2)$$

On the other hand we have for Bessel's function $J_i(x)$ the expression

$$J_i(x) = \sum_{m=0}^{m=\infty} (-1)^m \frac{\left[\frac{x}{2}\right]^{i+2m}}{m\,!\,(i+m)\,!}. \quad (3)$$

Therefore we can write

$$J_i(x) = \sum_{m=0}^{m=\infty} \frac{N_{-i,0,i+2m}}{(i+2m)\,!}\left[\frac{x}{2}\right]^{i+2m} \quad (4)$$

and, differentiating with regard to $x$,

$$\frac{dJ_i(x)}{dx} = \tfrac{1}{2} \sum_{m=0}^{m=\infty} \frac{N_{-i,0,i+2m}}{(i+2m-1)\,!}\left[\frac{x}{2}\right]^{i+2m-1}. \quad (5)$$

As an application of these formulas some properties of Bessel's functions

* Annals of Mathematics, Vol. 10, p. 1.

can be derived from the properties of Cauchy's numbers. Thus, for example, by (4)

$$J_{-i}(x) = \sum_{m=0}^{m=\infty} \frac{N_{i,0,-i+2m}}{(-i+2m)!} \left[\frac{x}{2}\right]^{-i+2m} \tag{6}$$

and, on account of

$$N_{-p,j,q} = (-1)^p N_{p,j,q}$$

formula (6) becomes

$$J_{-i}(x) = (-1)^i \sum_{m=0}^{m=\infty} \frac{N_{-i,0,-i+2m}}{(-i+2m)!} \left[\frac{x}{2}\right]^{-i+2m},$$

or, since $N_{-p,j,q} = 0$ for $-p+j+q < 0$, and therefore $N_{-i,0,-i+2m} = 0$ for $m < i$,

$$J_{-i}(x) = (-1)^i \sum_{m=i}^{m=\infty} \frac{N_{-i,0,-i+2m}}{(-i+2m)!} \left[\frac{x}{2}\right]^{-i+2m} = (-1)^i \sum_{m=0}^{m=\infty} \frac{N_{-i,0,2m+i}}{(2m+i)!} \left[\frac{x}{2}\right]^{2m+i},$$

that is to say

$$J_{-i}(x) = (-1)^i J_i(x).$$

In a similar way we can derive the well known relations

$$iJ_i(x) = \frac{x}{2}\left[J_{i-1}(x) + J_{i+1}(x)\right],$$

$$\frac{dJ_i(x)}{dx} = \tfrac{1}{2}\left[J_{i-1}(x) - J_{i+1}(x)\right],$$

from the following properties of Cauchy's numbers:

$$N_{-p,j+1,q} = N_{-p+1,j,q} + N_{-p-1,j,q}$$

$$N_{-p,j,q+1} = N_{-p+1,j,q} - N_{-p-1,j,q}.$$

As another application of our formulas let us express the function $\left[\frac{r}{a} - 1\right]^m$ of Celestial Mechanics by means of Bessel's functions given the formulas*

$$\left[\frac{r}{a} - 1\right]^m = \tfrac{1}{2} c_0^{(m)} + \sum_{i=1}^{i=\infty} c_i^{(m)} \cos i\zeta$$

$$c_i^{(m)} = (-1)^m \frac{2m}{i} \left[\frac{e}{2}\right]^m \sum_{k=0}^{k=\infty} \frac{N_{-i,m-1,i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k}. \tag{7}$$

* See *Tisserand*, Mécanique Céleste, tome I, pp. 237–239. For the values of $c_0^{(m)}$ see l. c. formulas (20) and (21). These values are also given below, namely, by formulas (9) and (10).

To this end we remark first that the number $N_{-p,j,q}$ can be expressed linearly in Cauchy's numbers of the form $N_{-p',0,q}$ by the formula*

$$N_{-p,j,q} = N_{-p+j,0,q} + \left[\frac{j}{1}\right] N_{-p+j-2,0,q} + \left[\frac{j}{2}\right] N_{-p+j-4,0,q} + \cdots + N_{-p-j,0,q},$$

which becomes in the present case

$$\begin{aligned} N_{-i,m-1,i-m+1+2k} = {} & N_{-i+m-1,0,i-m+1+2k} \\ & + \left[\frac{m-1}{1}\right] N_{-i+m-3,0,i-m+1+2k} \\ & + \left[\frac{m-1}{2}\right] N_{-i+m-5,0,i-m+1+2k} \\ & + \cdots\cdots\cdots\cdots \\ & + N_{-i-m+1,0,i-m+1+2k}. \end{aligned}$$

It must be noticed that in the first line of this formula $k \geqq 0$, in the second line $k \geqq 1$, in the third $k \geqq 2$, and so on; in the last line $k \geqq (m-1)$ because $N_{-p,j,q} = 0$ when $-p+j+q$ is negative. Hence

$$\begin{aligned} \sum_{k=0}^{k=r} \frac{N_{-i,m-1,i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k} = {} & \sum_{k=0}^{k=r} \frac{N_{-i+m-1,0,i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k} \\ & + \left[\frac{m-1}{1}\right] \sum_{k=1}^{k=r} \frac{N_{-i+m-3,0,i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k} \\ & + \left[\frac{m-1}{2}\right] \sum_{k=2}^{k=r} \frac{N_{-i+m-5,0,i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k} \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \\ & + \sum_{k=m-1}^{k=r} \frac{N_{-i-m+1,0,i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k} \\ = {} & \sum_{k=0}^{k=r} \frac{N_{-i+m-1,0,i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k} \\ & + \left[\frac{m-1}{1}\right] \sum_{k=0}^{k=r} \frac{N_{-i+m-3,0,i-m+3+2k}}{(i-m+2k+2)!} \left[\frac{ie}{2}\right]^{i-m+2k+2} \end{aligned}$$

* Annals of Mathematics, Vol. 10; l. c.

$$+\left\{\frac{m-1}{2}\right\}\sum_{k=0}^{k=r}\frac{N_{-i+m-5,\,0,\,i-m+5+2k}}{(i-m+2k+4)!}\left\{\frac{ie}{2}\right\}^{i-m+2k+4}$$

$$+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$

$$+\sum_{k=0}^{k=r}\frac{N_{-i-m+1,\,0,\,i+m+2k-1}}{(i+m+2k-2)!}\left\{\frac{ie}{2}\right\}^{i+m+2k-2}$$

$$=\frac{2}{i}\frac{d}{de}$$

$$\left\{J_{i-m+1}(ie)+\left\{\frac{m-1}{1}\right\}J_{i-m+3}(ie)+\left\{\frac{m-1}{2}\right\}J_{i-m+5}(ie)+\ldots+J_{i+m-1}(ie)\right\}$$

and therefore

$$c_i^{(m)}=(-1)^m\frac{4m}{i^2}\left\{\frac{e}{2}\right\}^m\frac{d}{de}$$

$$\left\{J_{i-m+1}(ie)+\left\{\frac{m-1}{1}\right\}J_{i-m+3}(ie)+\left\{\frac{m-1}{2}\right\}J_{i-m+5}(ie)+\ldots+J_{i+m-1}(ie)\right\}$$

which is the formula we intended to derive.

Conversely, suppose the function $\left\{\frac{r}{a}-1\right\}^m$ expressed with the help of Bessel's functions. To obtain this expression directly, we remark that

$$\left\{\frac{r}{a}-1\right\}^m=(-1)^m\left\{\frac{e}{2}\right\}^m 2\,.\,(2^{m-1}\cos^m u)$$

$$=(-1)^m\left\{\frac{e}{2}\right\}^m$$

$$2\left\{\cos mu+\left\{\frac{m}{1}\right\}\cos(m-2)u+\left\{\frac{m}{2}\right\}\cos(m-4)u+\ldots\right\}$$

the last term within the brackets being $\frac{1}{2}\left\{\frac{2m_1}{m_1}\right\}$ if $m=2m_1$; and $\left\{\frac{2m_1+1}{m_1}\right\}\cos u$ if $m=2m_1+1$. In either case the above development will contain a constant term which we will denote by $\frac{1}{2}c_0^{(m)}$. When $m=2m_1$

$$\tfrac{1}{2}c_0^{(m)}=(-1)^m\left\{\frac{e}{2}\right\}^m\left\{\frac{2m_1}{m_1}\right\}=\frac{(m_1+1)(m_1+2)\ldots 2m_1}{m_1!}\left\{\frac{e}{2}\right\}^{2m_1}\qquad(9)$$

when $m=2m_1+1$, since $\cos u=-\frac{1}{2}e+$ periodic terms, we shall have the constant term

$$\tfrac{1}{2}c_0^{(m)}=-\tfrac{1}{2}e(-1)^m\left\{\frac{e}{2}\right\}^m 2\left\{\frac{2m_1+1}{m_1}\right\}=2\,\frac{(m_1+2)(m_1+3)\ldots(2m_1+1)}{m_1!}\left\{\frac{e}{2}\right\}^{m+1}.\qquad(10)$$

Formulas (9) and (10) are identical with (20) and (21) of Tisserand (l. c. p. 238).

We have on the other hand for $m > 1$

$$\cos mu = m \sum_{i=1}^{i=\infty} [J_{i-m}(ie) - J_{i+m}(ie)] \frac{\cos i\zeta}{i},$$

which formula holds also for $m = 1$ if we leave out the non-periodic term. Hence

$$\left[\frac{r}{a} - 1\right]^m = \tfrac{1}{2} c_0^{(m)} + (-1)^m \left[\frac{e}{2}\right]^m 2 \sum_{i=1}^{i=\infty} A_i \frac{\cos i\zeta}{i}, \qquad (11)$$

where we have put

$$A_i = m \left\{ J_{i-m}(ie) - J_{i+m}(ie) \right\} + (m-2) \left[\frac{m}{1}\right] \left\{ J_{i-m+2}(ie) - J_{i+m-2}(ie) \right\}$$
$$+ (m-4) \left[\frac{m}{2}\right] \left\{ J_{i-m+4}(ie) - J_{i+m-4}(ie) \right\} + \ldots.$$

Now, it can be readily shown that

$$(m - 2p) \left[\frac{m}{p}\right] = m \left\{ \left[\frac{m-1}{p}\right] - \left[\frac{m-1}{p-1}\right] \right\}$$

so that

$$A_i = m \left\{ [J_{i-m}(ie) - J_{i-m+2}(ie)] + \left[\frac{m-1}{1}\right] [J_{i-m+2}(ie) - J_{i-m+4}(ie)] + \ldots \right\}$$
$$= \frac{2m}{i} \frac{d}{de} \left\{ J_{i-m+1}(ie) + \left[\frac{m-1}{1}\right] J_{i-m+3}(ie) + \ldots \right\},$$

and, therefore,

$$\left[\frac{r}{a} - 1\right]^m = \tfrac{1}{2} c_0^{(m)} + (-1)^m 4m \left[\frac{e}{2}\right]^m \sum_{i=1}^{i=\infty} \frac{\cos i\zeta}{i^2} \frac{d}{de}$$
$$\left\{ J_{i-m+1}(ie) + \left[\frac{m-1}{1}\right] J_{i-m+3}(ie) + \ldots \right\}$$

We have shown above that

$$\frac{d}{de} \left\{ J_{i-m+1}(ie) + \left[\frac{m-1}{1}\right] J_{i-m+3}(ie) + \ldots \right\}$$
$$= \frac{i}{2} \sum_{k=0}^{k=\infty} \frac{N_{-i,\,m-1,\,i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k}.$$

Hence, $c_i^{(m)}$ being given by formula (7) we shall have

$$\left[\frac{r}{a} - 1\right]^m = \tfrac{1}{2} c_0^{(m)} + \sum_{i=1}^{i=\infty} c_i^{(m)} \cos i\zeta$$

which solves the proposed problem.

NEW YORK, *July* 1, 1898.

# ON CIRCUIT INTEGRATION OVER A STRAIGHT LINE.

By Prof. W. H. Echols, Charlottesville, Va.

1. In the Annals of Mathematics, Vol. VIII, p. 137, I attempted to utilize Cauchy's method of integrating a function of a complex variable around a closed boundary containing a pole of the function, with the purpose in view of devising a similar method for a function of a real variable by integrating forward and backward to the place of beginning along the axis of the variable. I have since then seen the error in the fundamental formula in the paper referred to, and wish now to make that acknowledgment and to notice some interesting points in connection with the investigations connected with it.

2. We observe that $(t-a)^{-1}$ is the derivative of $\log(a-t)$ when $t<a$, and the derivative of $\log(t-a)$ when $t>a$. Hence

$$\int_{a-\alpha}^{a-\beta}\frac{dt}{t-a}=\log\frac{\beta}{\alpha};\qquad \int_{a+\beta}^{a+\alpha}\frac{dt}{t-a}=\log\frac{\alpha}{\beta}.$$

Therefore

$$\int_{a+\beta}^{a+\alpha}\frac{dt}{t-a}-\int_{a-\alpha}^{a-\beta}\frac{dt}{t-a}=\int_{a+\beta}^{a+\alpha}\frac{dt}{t-a}+\int_{a-\beta}^{a-\alpha}\frac{dt}{t-a},$$

$$=\log\left[\frac{\alpha}{\beta}\right]^2.$$

Let $q>p$ be two numbers defining an interval $((p,q))$ containing a number $a$. Let $\alpha$ and $\beta$ be two absolute numbers such that $\beta<\alpha$ and $a\pm\alpha$, $a\pm\beta$ are also in $((p,q))$. We call the indefinite *circuit* integral of a function over $((p,q))$, the sum of the four ordinary integrals taken from $p$ to $a-\alpha$, from $a+\beta$ to $q$, from $q$ to $a+\alpha$, from $a-\beta$ to $p$. Obviously, the intervals $((p,a-\alpha))$, $((a+\alpha,q))$ are passed over twice, once forward and once backward. The intervals $((a-\alpha,a-\beta))$, $((a+\beta,a+\alpha))$ are passed over once backward and forward respectively. While the gap $(a-\beta,a+\beta)$ is not entered upon at all by the variable.

The indefinite circuit integral of any function $y$ that is uniform and continuous everywhere in $((p,q))$ except at the point $a$, is evidently

$$\int_{(pq)}ydx=\int_{a-\beta}^{a-\alpha}ydx+\int_{a+\beta}^{a+\alpha}ydx.$$

The indefinite circuit integral of $(t-a)^{-1}$ over $((p, q))$ is, by the above, equal to $\log(\alpha/\beta)^2$. This is indefinite until the quotient $\alpha/\beta$ is assigned. We agree now once for all that the quotient $(\alpha/\beta)^2$ shall be $e$ for all values of $\alpha$ and $\beta$, and we let $\alpha$ and $\beta$ converge to zero. When this is the case, the limit to which converges the indefinite circuit integral as $\alpha$ and $\beta$ converge to zero is called the *definite* circuit integral with respect to $a$, or briefly the circuit integral with respect to $a$ over $((p, q))$. Thus

$$\int_{(pq)} \frac{dt}{t-a} = 1 .$$

3. Let $f(x)$ be any one-valued and continuous function at *all* points in $((p, q))$. Clearly, the circuit integral of $f(x)$ over $((p, q))$ is zero. But,

$$\int_{(pq)} \frac{f(t)}{t-a}\,dt = \int_{(pq)} \frac{f(t)-f(a)}{t-a}\,dt + f(a)\int_{(pq)} \frac{dt}{t-a},$$

$$= f(a), \tag{1}$$

if $f'(a)$ is an assigned number. Otherwise, since $t-a$ is always positive in $((a+\beta, a+\alpha))$ and negative in $((a-\alpha, a-\beta))$, we have for the circuit integral

$$\int_{(pq)} \frac{f(t)dt}{t-a} = \mathop{\mathrm{Lt}}_{\alpha=0} \left\{ f(t') \int_{a-\beta}^{a-\alpha} \frac{dt}{t-a} + f(t'') \int_{a+\beta}^{a+\alpha} \frac{dt}{t-a} \right\},$$

where $t'$ and $t''$ are numbers in $((a-\alpha, a-\beta))$, $((a+\beta, a+\alpha))$ respectively, and which converge to $a$ for $\alpha = 0$, while the limit of each of the integrals is $\log e^{\frac{1}{2}}$. Therefore the circuit integral is $f(a)$.

Let $a_1$ and $a_2$ be two points in $((p, q))$. The circuit integral with respect to $a_1$ and $a_2$ over $((p, q))$ is the sum of the circuit interval with respect to $a_1$ over $((p, r))$ and the circuit integral with respect to $a_2$ over $((r, q))$, where $r$ is any number between $a_1$ and $a_2$. The generalization with respect to $n$ points $a_1, \ldots, a_n$ in $((p, q))$ is obvious. We have

$$\int_{(pq)} \frac{f(t)dt}{(t-a_1)(t-a_2)} = \int_{(pr)} \frac{f(t)}{t-a_2}\frac{dt}{t-a_1} + \int_{(rq)} \frac{f(t)}{t-a_1}\frac{dt}{t-a_2},$$

$$= \frac{f(a_1)}{a_2-a_1} + \frac{f(a_2)}{a_1-a_2} = \frac{f(a_1)-f(a_2)}{a_1-a_2}. \tag{2}$$

In general, it follows at once that

$$\int_{(pq)} \frac{f(t)dt}{(t-a_1)\ldots(t-a_n)} = \sum_{r=1}^{n} \frac{f(a_r)}{\psi'(a_r)},$$

where $\phi'(t)$ is the derivative of $(t - a_1) \dots (t - a_n)$. If $f(x) = \phi'(x)$, this integral is $n$.

4. Again, by the above, we have

$$\int_{(pq)} \frac{f(t)dt}{(t - x)(t - a_1) \dots (t - a_n)} = \frac{f(x)}{\phi(x)} - \sum_1^n \frac{f(a_r)}{\phi'(a_r)} \frac{1}{x - a_r}. \tag{3}$$

Therefore

$$\phi(x) \int_{(pq)} \frac{f(t)dt}{(t - x)\phi(t)} = f(x) - \sum_1^n \frac{f(a_r)}{\phi'(a_r)} \frac{\phi(x)}{x - a_r}. \tag{4}$$

The function on the right vanishes when $x = a_1, \dots, a_n$, and the term under the sigma is a rational integral function of $x$ of degree $n - 1$. It is therefore equal to

$$\frac{(x - a_1) \dots (x - a_n)}{n!} f^n(u),$$

where $u$ is some number between the greatest and least of the numbers $x$, $a_1$, $\dots$, $a_n$, and we have

$$\int_{(pq)} \frac{f(t)dt}{(t - x)\phi(t)} = \frac{f^n(u)}{r!}, \tag{5}$$

provided $f(x)$ admits of differentiation and not unless.

If $f(x)$ is a rational integral function of degree less than $n$, the integral is zero.

5. Again, let

$$\Delta^n f(x) = \sum_{r=n}^0 (-1)^{n-r} C_{n,r} f(x + rh),$$

$$(x - a)^{(n+1)} = (x - a)(x - a - h) \dots (x - a - nh),$$

$x - a - rh$, $(r = 0, \dots, n)$, being points in $(p, q)$. Then

$$\int_{(pq)} \frac{f(t)dt}{(t - a)^{(n+1)}} = \sum_n^0 \frac{f(a + rh)}{(rh)(r - 1h) \dots h(-h) \dots (-r - nh)},$$

$$= \sum_n^0 (-1)^{n-r} \frac{n!}{r!(n - r)!} \frac{f(a + rh)}{h^n},$$

$$= \frac{1}{n!} \frac{\Delta^n f(a)}{h^n} = \frac{f^n(u)}{n!}, \tag{6}$$

when $u$ is between $a$ and $a + nh$.

6. Consider the integral (4) when $x = a$. Let $a_1, \ldots, a_n$ all converge to the limit $a$. Then since $u$ always lies between the greatest and least of these $n + 1$ numbers, it also has the limit $a$. The right side of (4) then has the limit $f^n(a)/n!$. We *define* this to be the value of the integral (4) when $a_1, \ldots, a_n$ are equal to $a$, and write in symbols

$$\int_{(pq)} \frac{f(t)dt}{(t-a)^{n+1}} = \frac{f^n(a)}{n!}. \tag{7}$$

It is to be distinctly understood that this symbol means that $a_1 \ldots, a_n$ are to be put equal to $a$ *after* the integration has been performed and not *before*.

The integral

$$\int_{(pq)} \frac{f(t)dt}{(t-a)^{n+1}}, \tag{8}$$

evaluated by the *ab initio* process of § 2 is *infinite* for all even integers $n$, and has no meaning. It is discarded from our consideration. For $n = 1$, however, we have by the *ab initio* process

$$\int_{(pq)} \frac{f(t)dt}{(t-a)^2} = \int_{(pq)} \left\{ \frac{f(t) - f(a)}{t-a} \frac{dt}{t-a} + f'(a) \frac{dt}{(t-a)^2} \right\}$$

$$= f'(a),$$

as before, since by § 1, we have the indefinite integral

$$\int_{(pq)} \frac{dt}{(t-a)^{m+1}} = \frac{1}{-m} \left\{ \left[ \frac{1}{\alpha^m} - \frac{1}{\beta^m} \right] + (-1)^m \left[ \frac{1}{\alpha^m} - \frac{1}{\beta^m} \right] \right\}, \tag{9}$$

$$= 0, \text{ when } m \text{ is odd},$$

$$= \frac{2}{m} \left[ \frac{1}{\beta^m} - \frac{1}{\alpha^m} \right], \text{ for } m \text{ even}.$$

The latter is therefore infinite when $\alpha = 0$.*

In order that (8) should be the same as (7), it is necessary and sufficient that (9) should be 0 for all values of the integer $m \neq 0$. This is not true for a real variable, as we have defined the integral, but is true for a complex variable under Cauchy's well known method.

* It does not seem possible to make this integral zero by letting $\alpha$ and $\beta$ converge to zero and their ratio to a finite real number different from 1.

With the definite meaning assigned to (7), however, we do have

$$\int_{(pq)} \frac{dt}{(t-a)^{n+1}} = 0$$

for all integers $n$, and equal to 1 for $n = 0$. Understanding clearly these conditions we can apply the process of circuit integration "across" any pole $a$ of multiplicity $n$, that pole being regarded as the result of running together of $n$ poles of order 1.

The applications of the method of circuit integration for a real variable are limited to the producing of certain finite-difference and differential-equations from the circuit integrals directly or from a given algebraic identity. Already we have had an illustration of this in the production of Lagrange's interpolation formula in (4) and of the result $\Delta^n f(x) = h^n f^n(u)$ of (6). This last lies at the foundation of the formation of differential equations from those in finite differences.

7. An important extension of (4) is as follows. If $\varphi(x) = (x-a_1)^{m_1+1} \dots (x-a_n)^{m_n+1}$, and $f(x)$ is any one-valued and continuous function in $((p, q))$. Then

$$\int_{(pq)} \frac{f(t)dt}{\varphi(t)} = \frac{f(x)}{\varphi(x)} - \sum_{r=1}^{n} \frac{1}{m_r!} \left[\frac{d}{dt}\right]^{m_r} \left\{ \frac{f(t)}{\varphi(t)} \frac{(t-a_r)^{m_r+1}}{x-t} \right\}_{t=a_r}. \tag{10}$$

Also

$$\left[\frac{d}{dt}\right]^{m_r} \left\{ \frac{f(t)}{\varphi(t)} \frac{(t-a_r)^{m_r+1}}{x-t} \right\} = \sum_{p=0}^{m_r} C_{m_r, p} \frac{(-1)^p p!}{(x-a_r)^{p+1}} \left[\frac{d}{dt}\right]^{m_r-p} \left\{ (t-a_r)^{m_r+1} \frac{f(t)}{\varphi(t)} \right\}_{t=a_r}.$$

If we multiply (10) by $\varphi(x)$ then the member on the right vanishes

$$N = m_1 + \dots + m_n + n$$

times at $a_1, \dots, a_n$, and since the rational integral function in $x$ under the sigma sign is of a degree less than $N$, we have

$$\varphi(x) \int_{(pq)} \frac{f(t)}{\varphi(t)} dt = \frac{\varphi(x)}{N!} f^N(u)$$

when $u$ lies between the greatest and least of $a_1, \dots, a_n, x$. If $f(x)$ is a rational integral function of degree less than $\varphi(x)$, the right side of (10) is zero, and we have the partition of the rational function into partial fractions.

8. Making use of Euler's identity in the form

$$\frac{1}{t-x} = \sum_{r=0}^{n} \frac{\psi_r(x)}{\psi_r(t)} \frac{1}{x-a_r} + \frac{\psi_n(x)}{\psi_n(t)} \frac{1}{t-x}, \tag{11}$$

where $\psi_r(x) = (x - a_0) \dots (x - a_r)$. Multiply this by $f(t)dt$ and take the circuit integral over an interval $((p, q))$ containing all the poles. The function $f(x)$ being as before one-valued and continuous in $((p, q))$, we have

$$f(x) = \sum_{r=0}^{n} \frac{\psi_r(x)}{\psi_r'(a_r)} \frac{f(a_r)}{x - a_r} + \psi_n(x) \int_{(pq)} \frac{f(t)\,dt}{(t - x)\,\psi_n(t)}. \tag{12}$$

The last term on the right is equal to

$$\frac{\psi_n(x)}{(n + 1)!} f^{n+1}(u),$$

$u$ between the greatest and least of the $a$'s and $x$.

If in this we put $a_0 = a$ and $a_p = a + ph$, and write $a$ before $(x - a)^{(0)} = 1$ and

$$(x - a)^{(n+1)} = (x - a)\,(x - a - h) \dots (x - a - nh),$$

(12) becomes

$$f(x) = \sum_{r=0}^{n} \frac{(x - a)^{(r)}}{r!} \Delta^r \frac{f(a)}{h^r} + \frac{(x - a)^{(n+1)}}{(n + 1)!} f^{n+1}(u). \tag{13}$$

If $h = 0$, (13) becomes

$$f(x) = \sum_{r=0}^{n} \frac{(x - a)^r}{r!} f^r(a) + \frac{(x - a)^{n+1}}{(n + 1)!} f^{n+1}(u), \tag{14}$$

provided $f(x)$ admits differentiation at $a$ and in $(ax)$.

9. This last result is also obtained by integrating the identity

$$\frac{1}{t - x} = \frac{1}{t - a} + \frac{x - a}{(t - a)^2} + \dots + \frac{(x - a)^n}{(t - a)^{n+1}} + \frac{(x - a)^{n+1}}{(t - x)\,(t - a)^n}, \tag{15}$$

after multiplying by $f(t)dt$, according to the meaning assigned in (7).

Again,

$$\int_{(pq)} \frac{f(t)dt}{(t - x)\,(t - a)^{n+1}} = \frac{1}{n!} \left[\frac{d}{da}\right]^n \frac{f(a)}{a - x} + \frac{f(x)}{(x - a)^{n+1}}$$

which produces (14) independently. Also this same integral can be written equal to

$$\int_{(pq)} \frac{f(t) - f(x)}{(t - x)\,(t - a)^{n+1}}\,dt + f(x) \int \frac{dt}{(t - x)\,(t - a)^{n+1}}$$

which is equal to

$$\frac{1}{n!} \left[\frac{d}{da}\right]^n \left\{\frac{f(x) - f(a)}{x - a}\right\},$$

giving another obvious identity.

10. If $\theta(x)$ $\theta_x$ be a function which does not vanish in $((p, q))$. Then

$$\int_{(pq)} \frac{dt}{(\theta_t - \theta_a)^{n+1}} = \int_{(pq)} \left[\frac{t-a}{\theta_t - \theta_a}\right]^{n+1} \frac{dt}{(t-a)^{n+1}}$$

$$= \frac{1}{n!}\left[\frac{d}{dt}\right]^n \left[\frac{t-a}{\theta_t - \theta_a}\right]^{n+1}_{t=a},$$

meaning $t = a$ after differentiation. Making use of the identity (15) in which we write $t = \theta_t$, $a = \theta_a$, then we can get Burmann's formula in a variety of forms. For example, multiply by $f'(t)dt$ and integrate (circuit) over $(ax)$ and we have

$$\frac{f'(x)}{\theta'(x)} = \sum_{r=0}^{n} \frac{(\theta_t - \theta_a)^r}{r!} D_t^r \left\{ \left[\frac{t-a}{\theta_t - \theta_a}\right]^{n+1} f'(t) \right\}_{t=a}$$

$$+ \frac{(\theta_x - \theta_a)^{n+1}}{(n+1)!} D_t^{n+1} \left\{ \left[\frac{t-a}{\theta_t - \theta_a}\right]^{n+1} \frac{f'(t)}{\theta_t - \theta_x} \right\}_{t=u}.$$

Multiply this by $\theta'(x)dx$ and integrate ordinarily from $a$ to $x$ and we get Burmann's with the remainder as a definite integral. Burmann's is produced directly with coefficients in a different form if we multiply (15) as above by $[f(t) - f(a)]\,\theta'(t)dt$ and circuit integrate over $(ax)$.

Circuit Integration renders no assistance in the expansion of functions further than producing the finite forms of the Differential Calculus which do not lead to general, but merely to particular, solutions of that problem.

# NOTE ON THE INVARIANT TOTAL DIFFERENTIAL EQUATION

$$Pdx + Qdy + Rdz = 0\,.$$

By Prof. J. M. Page, Charlottesville, Va.

If we write, as is usual,

$$\frac{dy}{dx} = y_1\,, \quad \frac{dz}{dx} = z_1\,,$$

Lie has shown that all equations of the general form

$$\Omega(x, y, z, y_1, z_1) = 0\,,$$

which are invariant under a given infinitesimal transformation

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}\,,$$

are found as follows. We form the expressions

$$\eta_1 = \frac{d\eta}{dx} - y_1 \frac{d\xi}{dx}\,, \quad \zeta_1 = \frac{d\zeta}{dx} - y_1 \frac{d\xi}{dx}\,;$$

and find the solutions of the linear partial differential equation of the first order in five variables,

$$U_1 f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta \frac{\partial f}{\partial z} + \eta_1 \frac{\partial f}{\partial y_1} + \zeta_1 \frac{\partial f}{\partial z_1} = 0\,.$$

Let these solutions be, respectively,

$$u(x, y, z), \quad v(x, y, z), \quad u_1(x, y, z, y_1, z_1)\,, \quad v_1(x, y, z, y_1, z_1)\,;$$

then the most general equation in the variables $x, y, z, y_1, z_1$, which is invariant under $Uf$ has the form

$$F(u, v, u_1, v_1) = 0\,.$$

Now if the functions $F$, $u_1$, and $v_1$ are so chosen that the equation $F = 0$ is *linear* in terms of $y_1$ and $z_1$, it is clear that we thus obtain a *total* differential equation in the variables $x$, $y$, $z$,—which is invariant under $Uf$.

As a simple example, it may be at once verified that the most general equation $F = 0$ which is invariant under the *perspective* transformation

$$Uf = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}\,,$$

has the form

$$F\left[\frac{y}{x}, \frac{z}{x}, y_1, z_1\right] = 0 .$$

Hence, all total differential equations of the form

$$P\left[\frac{y}{x}, \frac{z}{x}\right] dx + Q\left[\frac{y}{x}, \frac{z}{x}\right] dy + R\left[\frac{y}{x}, \frac{z}{x}\right] dz = 0 ,$$

that is, all *homogeneous* total differential equations are invariant under the perspective transformation.

In an analogous manner, we might proceed to establish an unlimited number of *types* of total differential equations, which are invariant under given infinitesimal transformations.

We shall now suppose that a total differential equation

$$Pdx + Qdy + Rdz = 0 \tag{1}$$

is given, and shall assume that this equation is invariant under a known infinitesimal transformation $Uf$. Two essentially different cases present themselves, according to whether the equation (1) satisfies the condition of integrability or not. We shall take up the latter case first.

I: If the invariant total differential equation (1) does not satisfy the condition of integrability, the *general* solution of this equation can always be obtained by a quadrature.

For the general solution of any non-integrable equation (1) is obtained by associating with (1) an equation of the form $\psi(x, y, z) = 0$, where $\psi$ is an arbitrary function. But since $u$ and $v$, the solutions of $Uf = 0$, are independent functions, it is readily seen that the general solution of (1) can always be obtained by associating with (1) an equation of the form

$$\Phi(u, v) = 0 ,$$

where $\Phi$ is an arbitrary function of $u$ and $v$. Hence we obtain a simultaneous system consisting of the equations (1) and

$$d\Phi - P'dx + Q'dy + R'dz = 0 ; \tag{2}$$

and this simultaneous system is, in turn, equivalent to the linear partial differential equation of the first order

$$Af - (QR' - Q'R)\frac{\partial f}{\partial x} + (RP' - R'P)\frac{\partial f}{\partial y} + (PQ' - P'Q)\frac{\partial f}{\partial z} = 0 .$$

Of course $Af = 0$ is invariant, since (1) and (2) are invariant; that is, the

equations $Af = 0$, $Uf = 0$ form a complete system with the known solution $\Phi(u, v)$.

Since the function $\Phi$ must contain two, at least, of the variables $x, y, z$,—we can introduce as new variables, $\Phi$ and two of the old variables,—say, $x, y$, $\Phi$. Also, since $A(\Phi) = U(\Phi) = 0$, it is clear that the equation $Af = 0$ becomes, in the new variables, a linear partial differential equation of the first order in $x$ and $y$, which is invariant under a known transformation $Uf$ in the same variables.* Hence, another solution of $Af = 0$ which is independent of $\Phi$,—that is, the general solution of (1) with the condition $\Phi = 0$, is obtained from an ordinary differential equation of the first order in two variables by a quadrature.

II: If the given invariant total differential equation

$$Pdx + Qdy + Rdz = 0 \tag{1}$$

satisfies the condition of integrability;—that is, if the general solution of (1) can be expressed by means of a single equation of the form

$$\Omega(x, y, z) = c,$$

then it is clear that $\Omega$, the integral-function of (1), will be at the same time the common solution of the equations

$$P\frac{\partial f}{\partial y} - Q\frac{\partial f}{\partial x} = 0, \quad P\frac{\partial f}{\partial z} - R\frac{\partial f}{\partial x} = 0,$$

which, in this case, form a complete system.

Since the family of surfaces $\Omega = c$ is invariant under $Uf$, the expression $U(\Omega)$ must be a function of $\Omega$ alone. If $U(\Omega) = 0$, the surfaces $\Omega = c$ are not transformed by $Uf$ at all; and, in this case, $Uf$ is said to be *trivial* with regard to equation (1). Hence, if we assume, as we always shall do, that $Uf$ is not *trivial*, the expression $U(\Omega)$ is not zero; and we can always choose the function $\Omega$ in such manner that $U(\Omega) = 1$.† Thus $\Omega$, the integral-function of (1), satisfies the three equations

$$Q\frac{\partial \Omega}{\partial x} - P\frac{\partial \Omega}{\partial y} = 0,$$

$$R\frac{\partial \Omega}{\partial x} - P\frac{\partial \Omega}{\partial z} = 0,$$

$$\xi\frac{\partial \Omega}{\partial x} + \eta\frac{\partial \Omega}{\partial y} + \zeta\frac{\partial \Omega}{\partial z} = 1.$$

* Lie's "Differentialgleichungen," p. 437.

† Cf. Lie.

Hence, it is immediately seen that $\Omega$ is obtained by a quadrature, in the form,

$$\Omega = \int \frac{Pdx + Qdy + Rdz}{P\xi + Q\eta + R\zeta}.$$

*That is, if* $P\xi + Q\eta + R\zeta \neq 0$, *the expression*

$$M = \frac{1}{P\xi + Q\eta + R\zeta}$$

*is an integrating factor of the invariant integrable equation* (1).

If $P\xi + Q\eta + R\zeta = 0$, it is clear that the path-curves of the transformation $Uf$ all lie *on* the surfaces $\Omega = c$; so that $Uf$ is *trivial* with regard to (1),—a case which was excluded.

We may remark that if the integral surfaces of the integrable invariant equation (1) are given in the form

$$\Phi(x, y, z, c) = 0, \qquad c = \text{const.}$$

an envelope of these $\infty^1$ surfaces may exist. This envelope must be a surface (possibly degenerated), which is clearly invariant under the transformation $Uf$. That is, the surface must be generated by path-curves of the transformation $Uf$, defined by the equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}. \tag{3}$$

Thus, since the equation to the envelope satisfies both the equations (3) and the equation (1), the envelope, (or the singular solution of (1)), is given by

$$\omega(x, y, z) = P\xi + Q\eta + R\zeta = 0. \tag{4}$$

*On* the envelope the integrating factor $M$ is, of course, infinite.

We may further notice that the directions of the characteristics of the envelope are given by the equation (1) in connection with the equation resulting from the total differentiation of the equation $\omega = \text{const.}$ That is, the characteristics of the envelope satisfy the equations

$$\begin{aligned} Pdx + Qdy + Rdz &= 0, \\ \omega_x dx + \omega_y dy + \omega_z dz &= 0; \end{aligned} \tag{5}$$

or the equivalent equations

$$\frac{dx}{Q\omega_z - R\omega_y} = \frac{dy}{R\omega_x - P\omega_z} = \frac{dz}{P\omega_y - Q\omega_x}. \tag{6}$$

If a cuspidal edge of the envelope $\omega = 0$ exists, it must clearly be a path-curve of the transformation $Uf$; that is, from (3) and (6), the cuspidal edge, if one exists, is given by the equations

$$\frac{\xi}{Q\omega_z - R\omega_y} = \frac{\eta}{R\omega_x - P\omega_z} = \frac{\zeta}{P\omega_y - Q\omega_x}.$$

It is easily seen that $\xi$, $\eta$, $\zeta$ cannot all vanish either at all points on the envelope, or at all points on the cuspidal edge (if these loci are not degenerated). Hence the envelope of the family of surfaces defined by the integrable total equation (1), which is invariant under a known infinitesimal transformation $Uf$, can be found without even a quadrature; and the same is true of the cuspidal edge of the envelope. The surfaces themselves can be found by a quadrature.

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